Proceedings of ICRC 2001: 1 © Copernicus Gesellschaft 2001

ICRC 2001

Fractional diffusion of cosmic rays

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Abstract.
We consider the propagation of galactic cosmic rays under assumption that the interstellar medium is a fractal one. An anomalous diffusion equation in terms of fractional derivatives is used to describe of cosmic ray propagation. The anomaly in used model results from large free paths ("Lévy flights") of particles between galactic inhomogeneities and long time a particle stays in a trap. An asymptotical solution of the anomalous diffusion equation for point instantaneous and impulse sources with inverse power spectrum relating to supernova bursts is found. It covers both subdiffusive and superdiffusive regimes and is expressed in terms of the stable distributions. The energy dependence of spectral exponent of observed particles in different regimes is discussed.

1 Introduction

A key to understanding the mechanism for cosmic-ray origin and acceleration is determination how cosmic-ray particles propagate through the interstellar medium (ISM). If the propagation process is determined by scattering at magnetic field inhomogeneities which have small-scale characters and can be considered as homogeneous Poisson ensemble, it can be described by a normal diffusion model (Ginzburg and Syrovatskii (1964); Berezinsky et al. (1990)). The normal diffusion process is characterised by a mean-squared displacement that increases with time, $\langle r^2(t)\rangle \propto t$, and by a Gaussian propagator.

However, during a few last decades, many evidences from both theory and observations of the existence of multiscale structures in the Galaxy have been found (see, for example, Lee and Jokipii (1976); Kaplan and Pikelner (1979); Lozinskaya (1986); Ruzmaikin et al. (1988); Vanshtein et al. (1989); Bochkarev (1990); Falgarone et al. (1991); Burlaga et al. (1993); Molchanov et al. (1993); Meish and Bally (1994); Armstrong et al. (1995); Minter and Spangler (1996); Cadavid et al. (1999)). Filaments, shells, clouds are entities widely spread in the ISM. A rich variety of structures that are created in interacting phases having different properties can be related to the fundamental property of turbulence called intermittency.

The stretching, bending and folding of magnetic field lines by turbulent motions of a medium partially coupled to the field make magnetic field also highly intermittent, especially on the smaller scales. As the turbulent zones do not fill space, the tool of the fractal geometry to characterise the ISM should be used. In such a fractal-like ISM we certainly do not expect the normal diffusion to hold.

Generalisation of this equation leads to anomalous diffusion (see the reviews by Bouchaud and Georges (1990); Isichenko (1992); West and Deering (1994) and Uchaikin and Zolotarev (1999)). In the case of anomalous diffusion $\langle r^2(t)\rangle \propto t^\gamma$, $t \to \infty$, where the exponent γ differs from 1, a value, that corresponds to the normal diffusion. Anomalous diffusion has a non-Gaussian propagator.

In our recent papers (Lagutin et al. (2000, 2001a,b)) we proposed an anomalous diffusion (superdiffusion) model for solution of the "knee" problem in primary cosmic-rays spectrum and explanation of different values of spectral exponent of protons and other nuclei at $E \sim 10^2 \div 10^5$ GeV/nucleon. The anomaly results from large free paths ("Lévy flights") of particles between magnetic domains—traps of the returned type. These paths are distributed according to inverse power law $\propto Ar^{-3-\alpha}$, $r \to \infty$, $\alpha \leq 2$ being an intrinsic property of fractal structures. We also assumed that the mean time a particle stays in a trap, $\langle\tau\rangle$, was finite.

In this paper we consider the propagation of galactic cosmic rays in fractal ISM without the latter assumption. We suppose now that the particle can spend anomalously a long time in the trap. An anomalously long time means that $\langle\tau\rangle = \int_0^\infty d\tau\, \tau\, q(\tau) = \infty$, so the distribution of the particles staying in traps, $q(\tau)$, has a tail of power-law type $\propto Bt^{-\beta-1}$, $t \to \infty$ with $\beta < 1$ ("Lévy trapping time").

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2 Particle diffusion in a fractal medium

After acceleration in a source the particle can be in one of two states: a state of “Lévy flights” or a state of rest-state of motion in a trap. Diffusion in this model is a process in which the particle state changes successively at random moments in time. Based on the continuous time random walk theory of Montroll and Weiss (1965) or on integral equation (see, for example, Uchaikin and Zolotarev (1999); Uchaikin (1999)) we can readily derive the fractional diffusion equation. The equation for Green’s function $G(\boldsymbol{r},t,E;E_0)$ without energy losses and nuclear interactions under condition that the particle started from origin $\boldsymbol{r_0}=0$ at time $t_0=0$ with energy E_0 has the form

$$\frac{\partial G}{\partial t}=-D(E,\alpha,\beta)D_{0+}^{1-\beta}(-\Delta)^{\alpha/2}G(\boldsymbol{r},t,E;E_0)$$

$$+\delta(\boldsymbol{r})\delta(t)\delta(E-E_0). \tag{1}$$

Here D_{0+}^{μ} denotes the Riemann-Liouville fractional derivative (Samko et al. (1987))

$$D_{0+}^{\mu}f(t)\equiv\frac{1}{\Gamma(1-\mu)}\frac{d}{dt}\int\limits_0^t(t-\tau)^{-\mu}f(\tau)\,d\tau,\quad \mu<1,$$

$(-\Delta)^{\alpha/2}$—the fractional Laplacian (called “Riss” operator (Samko et al. (1987)))

$$(-\Delta)^{\alpha/2}f(x)=\frac{1}{d_{m,l}(\nu)}\int\limits_{\mathrm{R}^m}\frac{\Delta_y^l f(x)}{|y|^{m+\nu}}dy,$$

where $l>\alpha$, $x\in\mathrm{R}^m$, $y\in\mathrm{R}^m$,

$$\Delta_y^l f(x)=\sum_{k=0}^{l}(-1)^k\binom{l}{k}f(x-ky)$$

and

$$d_{m,l}(\nu)=\int\limits_{\mathrm{R}^m}(1-e^{iy})^l|y|^{-m-\nu}dy.$$

The anomalous diffusivity $D(E,\alpha,\beta)$ is determined by the constants A and B in the asymptotic behaviour for “Lévy flights” (A) and “Lévy waiting time” (B) distributions:

$$D(E,\alpha,\beta)\propto A(\alpha)/B(E,\beta).$$

Taking into account that the probability to stay in a trap during the time interval t for particle with charge Z and mass number A depends on particle rigidity as $\propto R^{-\delta}$, we find $D\propto(E/Z)^{\delta}$.

The solutions of equations (1) with zero boundary conditions at infinity can be found by Laplace-Fourier transformations with use of formulae (Samko et al. (1987))

$$\int\limits_0^\infty e^{-\lambda t}D_{0+}^{\mu}G(\boldsymbol{r},t,E;E_0)\,dt$$

$$=\lambda^{\mu}\int_0^\infty e^{-\lambda t}G(\boldsymbol{r},t,E;E_0)\,dt=\lambda^{\mu}\tilde{G}(\boldsymbol{r},\lambda,E;E_0),$$

$$\int\limits_{\mathrm{R}^3}e^{i\boldsymbol{kr}}(-\Delta)^{\alpha/2}G(\boldsymbol{r},t,E;E_0)\,d\boldsymbol{r}$$

$$=|\boldsymbol{k}|^{\alpha}\int\limits_{\mathrm{R}^3}e^{i\boldsymbol{kr}}G(\boldsymbol{r},t,E;E_0)\,d\boldsymbol{r}=|\boldsymbol{k}|^{\alpha}\tilde{G}(\boldsymbol{k},t,E;E_0).$$

The Laplace-Fourier transformation solution of (1) is

$$\tilde{G}(\boldsymbol{k},\lambda,E;E_0)=\delta(E-E_0)\lambda^{\beta-1}$$

$$\times\int\limits_0^\infty\exp\left(-[\lambda^{\beta}+D(E,\alpha,\beta)|\boldsymbol{k}|^{\alpha}]y\right)dy.$$

Invert Laplace-Fourier transform we find

$$G(\boldsymbol{r},E,t;E_0)=\delta(E-E_0)\big(D(E_0,\alpha,\beta)t^{\beta}\big)^{-3/\alpha}$$

$$\times\Psi_3^{(\alpha,\beta)}\left(|\boldsymbol{r}|\big(D(E_0,\alpha,\beta)t^{\beta}\big)^{-1/\alpha}\right), \tag{2}$$

where

$$\Psi_3^{(\alpha,\beta)}(r)=\int\limits_0^\infty q_3^{(\alpha)}\left(r\tau^{\beta}\right)q_1^{(\beta,1)}(\tau)\,\tau^{3\beta/\alpha}\,d\tau. \tag{3}$$

Here $q_3^{(\alpha)}(r)=(2\pi)^{-3}\int\exp(-ikr-|k|^{\alpha})dk$ is the density of three-dimensional spherically-symmetrical stable distribution with characteristic exponent $\alpha\le 2$ (Zolotarev et al. (1999); Uchaikin and Zolotarev (1999)) and $q_1^{(\beta,1)}(t)$ is one-sided stable distribution with characteristic exponent β (Zolotarev (1983)):

$$q_1^{(\beta,1)}(t)=(2\pi i)^{-1}\int\limits_S\exp(\lambda t-\lambda^{\beta})d\lambda.$$

Let us remember that $q_3^{(2)}(r)$ is the normal (Gaussian) distribution density, $q_3^{(1)}(r)$ is the three-dimensional Cauchy density $[\pi(1+r^2)]^{-2}$, $q_1^{(1/2,1)}(r)$ is Lévy-Smirnov distribution. Other stable densities cannot be expressed through elementary functions, but there exist representations in terms of convergent and asymptotic series (Uchaikin and Zolotarev (1999)). Based on equation for Green’s function (1) it is easy to formulate the fractional diffusion equation for concentration:

$$\frac{\partial N}{\partial t}=-D(E,\alpha,\beta)D_{0+}^{1-\beta}(-\Delta)^{\alpha/2}N(\boldsymbol{r},t,E)$$

$$+S(\boldsymbol{r},t,E), \tag{4}$$

where $S(\boldsymbol{r},t,E)$ is a density function of sources distribution.

3 Spectra

Using Green's function (2) we can find the cosmic ray concentration for sources interesting for astrophysics. Thus, for point instantaneous source with inverse power spectrum

$$S(\boldsymbol{r},t,E)=S_0E^{-p}\delta(\boldsymbol{r})\delta(t)$$

we have

$$N(\boldsymbol{r},t,E)=S_0E^{-p}\big(D(E,\alpha,\beta)t^{\beta}\big)^{-3/\alpha}$$

$$\times\Psi_3^{(\alpha,\beta)}\left(r\big(D(E,\alpha,\beta)t^{\beta}\big)^{-1/\alpha}\right). \qquad (5)$$

For point impulse source

$$S(\boldsymbol{r},t,E)=S_0'E^{-p}\delta(\boldsymbol{r})\Theta(T-t)\Theta(t)$$

$$\Theta(\tau)=\begin{cases}1, & \tau>0,\\ 0, & \tau<0,\end{cases}$$

the solution is of the form

$$N(\boldsymbol{r},t,E)=\frac{S_0'E^{-p}}{D(E,\alpha,\beta))^{3/\alpha}}\int\limits_{\max[0,t-T]}^{t}\tau^{-3\beta/\alpha}$$

$$\times\Psi_3^{(\alpha,\beta)}\left(|\boldsymbol{r}|\big(D(E,\alpha,\beta)\tau^{\beta}\big)^{-1/\alpha}\right). \qquad (6)$$

Using the representation $N=N_0E^{-\eta}$ and the asymptotic behaviour of the scaling function $\Psi^{(\alpha,\beta)}(r)$ ($\alpha<2,\beta<1$)

$$\Psi_3^{(\alpha,\beta)}(r\to 0)\propto r^{-(3-\alpha)},$$

$$\Psi_3^{(\alpha,\beta)}(r\to\infty)\propto r^{-3-\alpha},$$

one can evaluate the variation of spectral exponent $\Delta\eta=\eta(E\to\infty)-\eta(E\to 0)$. It follows from (5) that

$$N(\boldsymbol{r},t,E)\propto E^{-p+\delta},\quad E\to 0,$$

$$N(\boldsymbol{r},t,E)\propto E^{-p-\delta},\quad E\to\infty,$$

$$\Delta\eta=2\delta.$$

In other words, in case of point instantaneous source in both subdiffusive ($\beta<\alpha/2$) and superdiffusive ($\beta>\alpha/2$) regimes the spectral exponent of observed particle increases with energy on 2δ, i.e. the cosmic ray spectrum steepens (the "knee").

The similar estimates for the impulse source (see (6)) give

$$N(\boldsymbol{r},t,E)\propto E^{-p+\delta},\quad E\to 0,$$

$$N(\boldsymbol{r},t,E)\propto E^{-p-\delta/\beta},\quad E\to\infty,$$

$$\Delta\eta=\delta(1+1/\beta).$$

Our analytical and numerical studies however show that this property of energy spectrum (the "knee") is lacking in the regimes of normal diffusion ($\alpha=2,\beta=1$) and subdiffusion ($\alpha=2,\beta<1$).

4 Conclusion

We considered the propagation of galactic cosmic ray in the fractal interstellar medium. Anomalous diffusion equation in terms of fractional derivates describing of cosmic ray propagation has been formulated. An asymptotical solution of this equation covered both subdiffusive and superdiffusive regimes has been expressed in terms of stable distributions.

Our results showed that the "knee" in the primary cosmic ray spectrum is due to anomalously large free paths ("Lévy flights") of particles, being an intrinsic property of the fractal interstellar medium.

This work was supported by RFBR grants 0001 00284 and 0002 17507 and program "Integration" (project 2.1–252).

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